

W -geometry and Isomonodromic Deformations^{1 2}

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Abstract

We introduce new times in the monodromy preserving equations. While the usual times related to the moduli of complex structures of Riemann curves such as coordinates of marked points, we consider the moduli of generalized complex structures (W -structures). We consider linear differential matrix equations depending on W -structures on an arbitrary Riemann curve. The monodromy preserving equations have a Hamiltonian form. They are derived via the symplectic reduction procedure from a free gauge theory as well as the associate linear problems. The quasi-classical limit of isomonodromy problem leads to integrable hierarchies of the Hitchin type. In this way the generalized complex structures parametrized the moduli of these hierarchies.

1 Introduction

The monodromy preserving equations arise from the consideration of a linear matrix equation on \mathbf{CP}^1

$$(\partial + A)\Psi = 0. \quad (1.1)$$

Assume for simplicity that meromorphic matrix function A has only the first order poles

$$A = \sum_{a=1}^n \frac{p_a}{z - x_a}.$$

The independence of the monodromies of Ψ around the poles on positions of the poles x_a gives rise to the monodromy preserving equations. In this case monodromy preserving equations is the Schlesinger system. This system is nontrivial if the number of poles $n > 3$. In particular, for $n = 4$ and for the two by two matrices the Schlesinger system is equivalent to the famous Painlevé VI equation. All these equations can be described as non autonomous Hamiltonian systems, where the role of times plays by the positions of poles. The positions of the poles correspond to some fixed complex structure on a sphere $\mathbf{CP}^1(n)$ with n the marked points. On the other hand the complex structure can be encoded in the Beltrami differential μ . It allows to deform the operator of complex structure $\bar{\partial} \rightarrow \bar{\partial} + \mu\partial$. The Beltrami differentials are defined up

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to diffeomorphisms of $\mathbf{CP}^1(n)$. Locally μ can be considered as an element of the Teichmüller space $H^1(\mathbf{CP}^1(n), \Gamma)$, ($\dim H^1(\mathbf{CP}^1(n), \Gamma) = n - 3$). Here Γ is the space of smooth sections of the tangent bundle $T\mathbf{CP}^1$. For example, the Beltrami differential can be chosen in the form

$$\mu = \sum_{a=1}^n (x_a - x_a^0) \bar{\partial} \chi_a(z, \bar{z}),$$

where $\chi_a(z, \bar{z})$ is a characteristic function of a small neighborhood of the fixed point x_a^0 (see (2.8)). In this way the Beltrami differentials are closely related to the geometry of $\mathbf{CP}^1(n)$.

There exists a generalization of this geometry that comes from the higher order differentials $\mu_j \in H^1(\mathbf{CP}^1(n), \Gamma^{\otimes j})$. Roughly speaking this generalization allows to deform the operator $\bar{\partial}$ as

$$\bar{\partial} + \sum_j \mu_j \partial^j.$$

The hidden geometry behind these deformations is known as W -geometry [1].

In this talk I present an attempt to introduce the times connected with μ_j , $j > 1$ in the Isomonodromic deformation problem in the similar way as it was done for the Beltrami differentials $\mu \sim \mu_1$. We start from the flat $\mathrm{SL}(N, \mathbf{C})$ -bundles over a Riemann curve $\Sigma_{g,n}$ of genus g with n marked points. It is a phase space for the standard Isomonodromic deformation problem on $\Sigma_{g,n}$ [2].

Then we consider the generalized deformations of complex structures on $\Sigma_{g,n}$ and the associated Isomonodromic deformation problems. The symplectic reduction procedure leads to meaningful equations and allows to write down the relevant linear systems.

The next point discussed here is the quasi-classical limit from the monodromy preserving equations to integrable hierarchies. It means in particular that in (1.1) the operator ∂ is replaced by $\kappa \partial$ and $\kappa \rightarrow 0$. In this case one comes from the Schlesinger system to the Gaudin system, where positions of the points are merely parameters [3]. The approach presented in [3] can be generalized on the Isomonodromic deformations with higher times. We can fix the higher order differentials up to the order $N - 1$. In this case the classical limit leads to the hierarchies of completely integrable hierarchies. When the W -moduli vanish they pass into the Hitchin systems [4, 5]. In this way the W -moduli can be considered as deformation parameters of integrable hierarchies.

2 Isomonodromy preserving equations as reduced Hamiltonian systems

1. The upstairs phase space \mathcal{R}^N . We consider a stable bundle $\mathrm{SL}(N, \mathbf{C})$ bundle E over a Riemann curve $\Sigma_{g,n}$ of genus g with n marked points. The upstairs phase space \mathcal{R}^N is the collection of the coadjoint generic orbits \mathcal{O}_a of $\mathrm{SL}(N, \mathbf{C})$ in the marked points x_a , $a = 1, \dots, n$ along with the connections

$$\nabla^{(1,0)} : \Omega^0(\Sigma_{g,n}, E) \rightarrow \Omega^{(1,0)}(\Sigma_{g,n}, E), \quad \nabla^{(0,1)} : \Omega^0(\Sigma_{g,n}, E) \rightarrow \Omega^{(0,1)}(\Sigma_{g,n}, E).$$

Over a small disk $\nabla^{(1,0)} = \kappa \partial + A$, $\nabla^{(0,1)} = \bar{\partial} + \bar{A}'$ and

$$\mathcal{R}^N = \{\kappa \partial + A, \bar{\partial} + \bar{A}'; (\mathcal{O}_1, \dots, \mathcal{O}_n)\}. \quad (2.1)$$

Here κ is the level. It will play the role of the Plank constant later, when we shall consider integrable theories arising in the classical limit ($\kappa \rightarrow 0$). The connections can have first order poles at the marked points.

Represent $p_a \in \mathcal{O}_a$ as $p_a = g_a p_a^0 g_a^{-1}$, $g_a \in \text{SL}(N, \mathbf{C})$ and p_a^0 fixes the conjugate class of \mathcal{O}_a . The Kirillov-Kostant symplectic form on \mathcal{O}_a in this parameterizations is

$$\text{tr}(d(p_a g_a^{-1}) dg_a) = \langle d(p_a g_a^{-1}) dg_a \rangle.$$

The symplectic form on the whole space \mathcal{R}^N is

$$\omega = \int_{\Sigma_{g,n}} \langle dA d\bar{A}' \rangle + \omega_{\mathcal{O}}, \quad \omega_{\mathcal{O}} = \sum_{a=1}^n \langle d(p_a g_a^{-1}) dg_a \rangle \quad (2.2)$$

The form is invariant under the action of the gauge group $\mathcal{G}_N = \{f\} \sim \Omega^0(\Sigma_g, \text{Aut} E)$

$$A \rightarrow f^{-1} \kappa \partial f + f^{-1} A f, \quad \bar{A}' \rightarrow f^{-1} \bar{\partial} f + f^{-1} \bar{A}' f, \quad (2.3)$$

$$p_a \rightarrow f_a^{-1} p_a f_a, \quad g_a \rightarrow g_a f_a, \quad f_a = f(z, \bar{z})|_{z=x_a}. \quad (2.4)$$

The corresponding vector fields are generated by the flatness constraint

$$\bar{\partial} A - \kappa \partial \bar{A}' + [\bar{A}', A] - 2\pi i \sum_{a=1}^n p_a \delta(x_a) = 0, \quad (2.5)$$

where $\delta(x_a)$ is a distribution defined by the integral

$$\frac{1}{2\pi i} \int_{\Sigma_{g,n}} \bar{\partial} \frac{1}{z - x_a},$$

z is a local coordinate in a neighborhood of x_a .

2. Hamiltonians and times. The complex structure on $\Sigma_{g,n}$, defined locally by the $\bar{\partial}$ operator, can be deformed by the Beltrami differential $\mu \in \Omega^{(-1,1)}(\Sigma_{g,n})$:

$$\bar{\partial} \rightarrow \bar{\partial} + \mu \partial, \quad \mu = \mu(z, \bar{z}) \frac{\partial}{\partial z} \otimes d\bar{z}.$$

It should be mentioned that μ depends on the choice of local coordinates as

$$\mu(w, \bar{w}) = \frac{-\mu(z, \bar{z}) \partial w + \bar{\partial} w}{\mu(z, \bar{z}) \partial \bar{w} - \bar{\partial} \bar{w}}, \quad w = w(z, \bar{z}).$$

In particular, if

$$w = z - \epsilon(z, \bar{z}), \quad \bar{w} = \bar{z}$$

and $\epsilon(z, \bar{z})$ is small, then

$$\mu = \bar{\partial} \epsilon(z, \bar{z}).$$

If $w(z, \bar{z})$ is a global diffeomorphism than $\mu(w, \bar{w})$ is equivalent to $\mu(z, \bar{z})$. The equivalence relations in $\Omega^{(-1,1)}(\Sigma_{g,n})$ is the moduli space of complex structures on $\Sigma_{g,n}$. The tangent space to the moduli space is the Teichmüller space $\mathcal{T}_{g,n} \sim H^1(\Sigma_{g,n}, \Gamma)$, where $\Gamma \in T\Sigma_g$. From the Riemann-Roch theorem one has

$$l = \dim \mathcal{T}_{g,n} = 3(g-1) + n. \quad (2.6)$$

Let $(\mu_1^0, \dots, \mu_l^0)$ be the basis in the vector space $H^1(\Sigma_{g,n}, \Gamma)$. We use the same notation μ for the elements from $H^1(\Sigma_{g,n}, \Gamma)$. Then

$$\mu = \sum_{s=1}^l t_s \mu_s^0. \quad (2.7)$$

In particular, for the moduli related to the marked points one can choose a basis in the following form. Let (z, \bar{z}) be the local coordinates in a neighborhood of the marked point x_a and \mathcal{U}_a is a neighborhood of x_a such that $x_b \notin \mathcal{U}_a$ if $x_b \neq x_a$. Define a C^∞ function

$$\chi_a(z, \bar{z}) = \begin{cases} 1, & z \in \mathcal{U}'_a \subset \mathcal{U}_a \\ 0, & z \notin \mathcal{U}_a. \end{cases} \quad (2.8)$$

Then in (2.7)

$$\mu_a = t_a \bar{\partial} \chi_a(z, \bar{z}), \quad (\mu_a^0 = \bar{\partial} \chi_a(z, \bar{z}), \quad t_a = x_a^0 - x_a). \quad (2.9)$$

We replace \bar{A}' by \bar{A}

$$\bar{A} = \bar{A}' + \frac{1}{\kappa} \mu A. \quad (2.10)$$

The gauge transformations (2.3) acts on \bar{A} as

$$\bar{A} \rightarrow f^{-1}(\bar{\partial} + \mu \partial) f + f^{-1} \bar{A} f. \quad (2.11)$$

The choice of complex structure defines the polarization (A, \bar{A}) of \mathcal{R}^N . Thus, we define the bundle \mathcal{P}_1^N over $\mathcal{T}_{g,n}$ with the local coordinates

$$(A, \bar{A}, \mathbf{p}, \mathbf{t}), \quad \mathbf{p} = (p_1, \dots, p_n), \quad \mathbf{t} = (t_1, \dots, t_l).$$

$$\begin{array}{ccc} \mathcal{P}_1^N & & \\ \downarrow & \mathcal{R}^N & \\ \mathcal{T}_{g,n} & & \end{array}$$

The bundle \mathcal{P}_1^N plays the role the extended phase space, while \mathcal{R}^N is the standard phase space with non degenerate form (2.2) and t_1, \dots, t_l are the times. Substitute $\bar{A}' = \bar{A} - \frac{1}{\kappa} \mu A$ in the symplectic form (2.2). Then it takes the standard form on the extended phase space [6]

$$\omega = \omega_0 - \frac{1}{\kappa} \sum_{s=1}^l dH_s dt_s, \quad (2.12)$$

where

$$\omega_0 = \int_{\Sigma_{g,n}} \langle dA \, d\bar{A} \rangle + \omega_{\mathcal{O}}, \quad (2.13)$$

and

$$H_s = \frac{1}{2} \int_{\Sigma_{g,n}} \langle A^2 \rangle \mu_s^0. \quad (2.14)$$

The symplectic form ω is defined on the total space of \mathcal{P}_1^N . It is degenerate on l vector fields D_s : $\omega(D_s, \cdot) = 0$, where

$$D_s = \partial_{t_s} + \frac{1}{\kappa} \int_{\Sigma_{g,n}} \langle A \frac{\delta}{\delta \bar{A}} \rangle \mu_s^0 = \partial_{t_s} + \frac{1}{\kappa} \{H_s, \cdot\}_{\omega_0},$$

and the Poisson brackets are inverse to the non-degenerate form ω_0 on the fibers. The vector fields D_s define the equations of motion for any function f on \mathcal{P}_1^N

$$\frac{df}{dt_s} = \partial_{t_s} f + \frac{1}{\kappa} \{H_s, f\}_{\omega_0}.$$

In addition, there are the consistency conditions for the Hamiltonians (the Whitham equations [7])

$$\kappa \partial_s H_r - \kappa \partial_r H_s + \{H_r, H_s\} = 0. \quad (2.15)$$

Since the Hamiltonians (2.14) commute there exists the tau-function

$$H_s = \frac{\partial}{\partial t_s} \log \tau, \quad \tau = \exp \frac{1}{2} \sum_{s=1}^l \int_{\Sigma_{g,n}} < A^2 > \mu_s.$$

In particular,

$$\partial_s A = 0, \quad \partial_s \bar{A} = \frac{1}{\kappa} A \mu_s^0, \quad \partial_s p_a = 0 \quad (2.16)$$

and therefore

$$A = A_0, \quad \bar{A} = \bar{A}_0 + \frac{1}{\kappa} \mu A_0.$$

Let $\Psi \in \Omega^{(0)}(\Sigma_{g,n}, \text{Aut} P)$ be a solution of the linear system

$$(\kappa \partial + A) \Psi = 0, \quad (2.17)$$

$$(\bar{\partial} + \sum_{s=1}^l t_s \mu_s^0 \partial + \bar{A}) \Psi = 0, \quad (2.18)$$

$$\kappa \partial_s \Psi = 0, \quad (\partial_s = \partial_{t_s}). \quad (2.19)$$

The monodromy of Ψ is the transformation

$$\Psi \rightarrow \Psi \mathcal{Y}, \quad \mathcal{Y} \in \text{Rep}(\pi_1(\Sigma_{g,n}) \rightarrow \text{SL}(N, \mathbb{C})).$$

The equation (2.19) means that the monodromy is independent on the times. The equations of motion (2.16) for A and \bar{A} are the consistency conditions (2.17),(2.19) and (2.18),(2.19) correspondingly. The consistency condition of (2.17) and (2.18) is the flatness constraint (2.5).

3. Symplectic reduction. Up to now the equations of motion, the linear problem, and the tau function are trivial. The meaningful equations arise after the gauge fixing with respect to (2.3),(2.4) and imposing the corresponding constraints (2.5). The set \mathcal{R}_{red}^N of the gauge orbits on the constraint surface (2.5) is the moduli space of flat connections

$$\mathcal{R}_{red}^N = (2.5)/\mathcal{G}_N = \mathcal{R}^N // \mathcal{G}_N.$$

Let us fix \bar{A} :

$$\bar{L} = f^{-1}(\bar{\partial} + \mu \partial) f + f^{-1} \bar{A} f. \quad (2.20)$$

For stable bundles \bar{L} can be choose in a such way, that $\bar{L} = 0$ for $g = 0$, \bar{L} =diagonal constant matrix for $g = 1$, and at least antiholomorphic for $g > 1$. Then the dual field

$$L = f^{-1} \kappa \partial f + f^{-1} A f \quad (2.21)$$

can be found from the moment equation (2.5)

$$(\bar{\partial} + \partial \mu) L - \kappa \partial \bar{L} + [\bar{L}, L] = 2\pi i \sum_{a=1}^n p_a \delta(x_a). \quad (2.22)$$

Here we preserve the notion p_a for the gauge transformed element of \mathcal{O}_a . The gauge fixing (2.20) and the moment constraint (2.22) kill almost all degrees of freedom. The fibers $\mathcal{R}^{red} = \{L, \bar{L}, \mathbf{p}\}$ become finite-dimensional, as well as the bundle $\mathcal{P}_1^{red, N}$:

$$\dim \mathcal{R}_{red}^N = 2(N^2 - 1)(g - 1) + N(N - 1)n, \quad (2.23)$$

$$\dim \mathcal{P}_{1\ red}^N = (2N^2 + 1)(g - 1) + (N^2 - N + 1)n.$$

On $\mathcal{P}_{1\ red}^N$ ω (2.12) preserves its form

$$\omega_0 = \int_{\Sigma_{g,n}} \langle dL \ d\bar{L} \rangle + \omega_{\mathcal{O}}, \quad H_s = H_s(L) = \frac{1}{2} \int_{\Sigma_{g,n}} \langle L^2 \rangle \mu_s^0. \quad (2.24)$$

But now, due to (2.22), the system is no long free because L depends on \bar{L} , \mathbf{p} . Moreover, because L depends explicitly on \mathbf{t} , the system (2.24) is non autonomous.

Let $M_s = \partial_s f f^{-1}$. Then the equations of motion on \mathcal{R}_N (2.16) take the form

$$\kappa \partial_s L - \kappa \partial M_s + [M_s, L] = 0, \quad s = 1, \dots, l, \quad (2.25)$$

$$\kappa \partial_s \bar{L} - (\bar{\partial} + \partial \mu) M_s + [M_s, \bar{L}] = L \mu_s^0. \quad (2.26)$$

The equations (2.25) are the analog of the Lax equations. The essential difference is the differentiation ∂ with respect to the spectral parameter. The equation (2.26) allows define M_s . These equations reproduce the Schlesinger system, Elliptic Schlesinger system, multi-component generalization of the Painlevé VI equation [2]. The equations (2.25), (2.26) along with (2.22) are consistency conditions for the linear system

$$(\kappa \partial + L) \Psi = 0, \quad (2.27)$$

$$(\bar{\partial} + \sum_s t_s \mu_s^0 \partial + \bar{L}) \Psi = 0, \quad (2.28)$$

$$(\kappa \partial_s + M_s) \Psi = 0, \quad (s = 1, \dots, l_2). \quad (2.29)$$

The equations (2.29) provides the isomonodromy property of the system (2.27), (2.28) with respect to variations of the times t_s . For this reason we call the nonlinear equations (2.25) the Hierarchy of the Isomonodromic Deformations.

4. Scaling limit. Consider the limit $\kappa \rightarrow 0$. The value $\kappa = 0$ is called critical. The symplectic form ω (2.12) is singular in this limit. Let us replace the times

$$t_s \rightarrow t_s^0 + \kappa t_s^H, \quad (t_s^H - \text{Hitchin times})$$

and assume that the times t_s^0 , ($s = 1, \dots, l$) are fixed. After this rescaling the form (2.12) becomes regular. The rescaling procedure means that we blow up a vicinity of the fixed point $\mu(0) = t_s^0 \mu_s^{(0)}$ in $\mathcal{T}_{g,n}$ and the whole dynamic is developed in this vicinity. This fixed point is defined by the complex coordinates

$$w_0 = z - \sum_s t_s^0 \epsilon_s(z, \bar{z}), \quad \bar{w}_0 = \bar{z}, \quad \partial_{\bar{w}_0} = \bar{\partial} + \mu(0) \partial. \quad (2.30)$$

For $\kappa = 0$ the connection A is transformed into the one-form Φ (the Higgs field) $\kappa \partial + A \rightarrow \Phi$, (see (2.3)). Let $L^0 = \lim_{\kappa \rightarrow 0} L$, $\bar{L}^0 = \lim_{\kappa \rightarrow 0} \bar{L}$. Then we obtain the autonomous Hamiltonian systems with the form

$$\omega^H = \int_{\Sigma_{g,n}} \langle dL^0 \ d\bar{L}^0 \rangle + 2\pi i \sum_{a=1}^n \langle d(p_a g_a^{-1}) \ dg_a \rangle$$

and the commuting time independent quadratic integrals

$$H_s = \frac{1}{2} \int_{\Sigma_{g,n}} < L_0^2 > \mu_s^0.$$

The phase space \mathcal{R}_{red}^N turns into the cotangent bundle to the moduli of stable holomorphic $\mathrm{SL}(N, \mathbf{C})$ -bundles over $\Sigma_{g,n}$.

The corresponding set of linear equations has the following form. The level κ can be considered as the Planck constant (see (2.27)). We consider the quasi-classical regime

$$\Psi = \phi \exp \frac{\mathcal{S}}{\kappa}, \quad (2.31)$$

where ϕ is a group-valued function and \mathcal{S} is a scalar phase. Assume that

$$\frac{\partial}{\partial \bar{w}_0} \mathcal{S} = 0, \quad \frac{\partial}{\partial t_s^H} \mathcal{S} = 0. \quad (2.32)$$

In the quasi-classical limit we set

$$\partial_{w_0} \mathcal{S} = \lambda. \quad (2.33)$$

Define the Baker-Akhiezer function

$$Y = \phi \exp \sum_{s=1}^l t_s^H \frac{\partial \mathcal{S}}{\partial t_s^0}.$$

Then instead of (2.27), (2.28), (2.29) we obtain in the limit $\kappa \rightarrow 0$

$$\begin{aligned} (\lambda + L^0)Y &= 0, \\ (\bar{\partial}_{\bar{w}_0} + \lambda \sum_{s=1}^l t_s \mu_s^0 + \bar{L}^0)Y &= 0, \\ (\partial_s + M_s^0)Y &= 0, \quad (s = 1, \dots, l). \end{aligned}$$

Note, that the consistency conditions for the first and the last equations are the standard Lax equations

$$\partial_s L^0 + [M_s^0, L^0] = 0, \quad (2.34)$$

while the consistency conditions for the first and the second equations is just the Hitchin equation, defining L^0

$$\partial_{\bar{w}_0} L^0 + [\bar{L}^0, L^0] = 2\pi i \sum_{a=1}^n \delta(x_a) p_a. \quad (2.35)$$

It follows from (2.34) that the resulting Hamiltonian system is completely integrable [4, 5]. The commuting integrals are

$$H_{s,l} = \frac{1}{j+1} \int_{\Sigma_{g,n}} < L_0^{j+1} > \mu_{s,j}^0, \quad (j = 1, \dots, N-1).$$

The differentials $\mu_{s,j}^0$ will be defined in next Section.

The gauge properties of the Higgs field allows to define the spectral curve

$$\mathcal{C} : \det(\lambda + L) = 0.$$

The fixed times t_s^0 change the complex structure of the spectral curve. The one-form $\partial_{w_0} \mathcal{S}$ (2.33) plays the role of the Seiberg-Witten differential on \mathcal{C} (2.33). It is meromorphic on \mathcal{C} in terms of the deformed complex structure (2.32).

3 Flat bundles and higher times

1. Higher times. The higher times μ_j are related to the differentials $\Omega^{(-j,1)}(\Sigma_{g,n})$. They define generalized deformations of the operator $\bar{\partial}$ by adding the higher order operators ∂^j (see below (3.14),(3.20)). There is an equivalence relations provided by a groupoid action on $\Omega^{(-j,1)}(\Sigma_{g,n})$ [8]. The space of orbits are the moduli space of generalized complex structures $\mathcal{M}_{g,n}^{(j)}$. The tangent space $\mathcal{T}_{g,n}^{(j)}(\Sigma_{g,n})$ to $\mathcal{M}_{g,n}^{(j)}$ at $\mu_j = 0$ is isomorphic to $H^1(\Sigma_{g,n}, \Gamma^{\otimes j})$. It has dimension

$$l_j = \dim H^1(\Sigma_{g,n}, \Gamma^{\otimes j}) = (2j+1)(g-1) + jn, \quad (3.1)$$

($l_1 = l$ (2.6)). Let $(\mu_{1,j}^{(0)}, \dots, \mu_{l_j,j}^{(0)})$ be the basis of $\mathcal{T}_{g,n}^{(j)}$. Locally

$$\mu_{s,j}^{(0)} = \mu_{s,j}^{(0)}(z, \bar{z}) \left(\frac{\partial}{\partial z} \right)^j \otimes d\bar{z}.$$

Then $\mu_j \in \mathcal{T}_{g,n}^{(j)}$ is represented as

$$\mu_j = \sum_{s=1}^{l_j} t_{s,j} \mu_{s,j}^{(0)}. \quad (3.2)$$

The parameters related to the marked points can be chosen in the form (see (2.9))

$$\mu_a = \sum_{s=0}^{j-1} t_{s,a} (z - x_a)^s \bar{\partial} \chi_a(z, \bar{z}). \quad (3.3)$$

They cover $\mathcal{T}_{g,n}^{(j)}$ for the rational curves $\mathbf{CP}^1(n)$. The whole set of times is the linear space

$$\mathcal{T}(k)_{g,n} = \oplus_{j=1}^k \mathcal{T}_{g,n}^{(j)}.$$

For $k=1$ it is the Teichmüller space defined above. From (3.1)

$$l(k) = \dim \mathcal{T}(k)_{g,n} = ((k+1)^2 - 1)(g-1) + \frac{k(k+1)}{2}n \quad (3.4)$$

2. Extended phase space. We will use the same phase space \mathcal{R}^N (2.1) and ω (2.2). To introduce the higher times consider an operator $\nabla^{(j,0)}$ acting from $\Omega^0(\Sigma_{g,n}, E) \rightarrow \Omega^{(j,0)}(\Sigma_{g,n}, E)$. Locally, over a small disk it takes the form

$$\nabla^{(j,0)} = (\kappa \partial + A)^j.$$

Then represent $\nabla^{(0,1)} \sim \bar{\partial} + \bar{A}'$ as

$$\bar{\partial} + \bar{A}' = \bar{\partial} + \bar{A} - \frac{1}{\kappa} \sum_{j=1}^k \mu_j (\kappa \partial + A)^j.$$

Here \bar{A} is a new field related to the initial fields A, \bar{A}' as

$$\bar{A} = \bar{A}' + \frac{1}{\kappa} \sum_{j=1}^k \mu_j \tilde{A}^{(j)} = \bar{A}' + \frac{1}{\kappa} \sum_{j=1}^k \tilde{A}^{(j)} \sum_{s=1}^{l_j} t_{s,j} \mu_{s,j}^{(0)}, \quad (3.5)$$

where $\tilde{A}^{(j)}$ are defined by the recurrence relation

$$\tilde{A}^{(0)} = 1, \quad \tilde{A}^{(j)} = (\kappa\partial + A)\tilde{A}^{(j-1)}.$$

Generically

$$\tilde{A}^{(j)} = A^j + \kappa \sum_{m=1}^{j-1} mA^{m-1}\partial AA^{j-m-1} + \dots + (\kappa\partial)^{j-1}A. \quad (3.6)$$

In particular,

$$\begin{aligned} \tilde{A}^{(1)} &= A, & \tilde{A}^{(2)} &= A^2 + \kappa\partial A, \\ \tilde{A}^{(3)} &= A^3 + \kappa(\partial AA + 2A\partial A) + \kappa^2\partial^2 A. \end{aligned}$$

The extended phase space \mathcal{P}_k^N is a bundle over $\mathcal{T}(k)_{g,n}$

$$\begin{array}{ccc} \mathcal{P}_k^N & & \\ \downarrow & \mathcal{R}^N & \\ \mathcal{T}(k)_{g,n} & & \end{array}$$

with the local coordinates

$$(A, \bar{A}, \mathbf{p}, \mathbf{t}), \quad \mathbf{p} = (p_1, \dots, p_n), \quad \mathbf{t} = (t_{s,j}), \quad s = 1, \dots, l_j, \quad j = 1, \dots, k.$$

The symplectic form ω (2.2) gives rise to the degenerate symplectic form on \mathcal{P}_k^N . In terms of A and \bar{A} (2.2) is equal

$$\omega = \int_{\Sigma_{g,n}} \langle dA \, d\bar{A} \rangle + \omega_{\mathcal{O}} - \frac{1}{\kappa} \int_{\Sigma_{g,n}} \langle dA \, d(\sum_{j=1}^k \tilde{A}^{(j)} \mu_j) \rangle. \quad (3.7)$$

For $k > 1$ the form cannot be represented in the Hamiltonian form as before (2.12). Thus, the notion of the Whitham equations (2.15) and the tau-function become obscure. On the other hand ω is degenerate on the $l(k)$ vector fields

$$D_{(s,j)} = \kappa\partial_{t_{s,j}} + \mu_{s,j}^0 \tilde{A}^{(j)} \frac{\delta}{\delta \bar{A}}. \quad (3.8)$$

The equations of motion for any observable $f(A, \bar{A}, \mathbf{t})$ takes the form

$$\partial_{t_{s,j}} f(A, \bar{A}, \mathbf{t}) = D_{(s,j)} f(A, \bar{A}, \mathbf{t}).$$

In particular, for A and \bar{A} one has

$$\partial_{t_{s,j}} A = 0, \quad (3.9)$$

$$\partial_{t_{s,j}} \bar{A} = \frac{1}{\kappa} \mu_{s,j}^0 \tilde{A}^{(j)}. \quad (3.10)$$

It follows from (3.6) that the gauge action \mathcal{G}_N on A (2.3) induces the following transformations of $\tilde{A}^{(j)}$

$$\tilde{A}^{(j)} \rightarrow f^{-1} \left(\sum_{i=1}^j \kappa^{j-i} C_j^i \tilde{A}^{(i)} \partial^{j-i} \right) f, \quad \left(C_j^i = \frac{j!}{i!(j-i)!} \right).$$

Then for the new field \bar{A}

$$\bar{A} \rightarrow f^{-1} \left(\bar{\partial} + \sum_{i=1}^k \mu_i \sum_{l=0}^{i-1} \kappa^{i-l-1} C_i^l \tilde{A}^{(l)} \partial^{i-l} \right) f + f^{-1} \bar{A} f. \quad (3.11)$$

The constraints, generating these transformations, are read off from (2.5) and (3.5)

$$\bar{\partial}A + \sum_{j=1}^k \partial(\mu_j \tilde{A}^{(j)}) - \kappa \partial \bar{A} + [\bar{A} - \frac{1}{\kappa} \sum_{j=1}^k \mu_j \tilde{A}^{(j)}, A] - 2\pi i \sum_{a=1}^n p_a \delta(x_a) = 0. \quad (3.12)$$

But, now the constraints become nonlinear (see (3.6)) As for the case $k = 1$ the equations of motion (3.9), (3.10) and the constraints (3.12) are the consistency conditions for the linear system (2.17), (2.18) and (2.19). In general case they take the form

$$(\kappa \partial + A)\Psi = 0, \quad (3.13)$$

$$\left(\bar{\partial} + \frac{1}{\kappa} \sum_{j=1}^k \mu_j \sum_{m=0}^{j-1} C_j^m \tilde{A}^{(m)} (\kappa \partial)^{j-m} + \bar{A} \right) \Psi = 0, \quad (3.14)$$

$$\kappa \partial_{s,j} \Psi = 0, \quad (\partial_s = \partial_{t_{s,j}}). \quad (3.15)$$

The monodromy of (3.13) and (3.14) is independent of the whole set of times $\mathbf{t} \in \mathcal{T}(k)$.

Consider for example the case $k = 2$. The previous expressions take the form

$$\begin{aligned} \omega = \int_{\Sigma_{g,n}} < dA (d\bar{A} - \partial dA \sum_{s=1}^{l_2} t_{s,2} \mu_{s,2}^{(0)}) > + \omega_{\mathcal{O}} - \frac{1}{2\kappa} \int_{\Sigma_{g,n}} \left(< dA^2 > \sum_{s=1}^{l_1} \mu_{s,1}^{(0)} \right) dt_{s,1} \\ - \frac{1}{3\kappa} \int_{\Sigma_{g,n}} \left(< dA(A^2 - \kappa \partial A) > \sum_{s=1}^{l_2} \mu_{s,2}^{(0)} \right) dt_{s,2}. \end{aligned} \quad (3.16)$$

The time evolutions of \bar{A} are defined as

$$\partial_{t_{s,1}} \bar{A} = \frac{1}{\kappa} \mu_{s,1}^0 A, \quad \partial_{t_{s,2}} \bar{A} = \frac{1}{\kappa} \mu_{s,2}^0 (A^2 + \kappa \partial A). \quad (3.17)$$

The gauge symmetries preserving ω_0 for $k = 2$ have the form

$$\bar{A} \rightarrow f^{-1} \left(\bar{\partial} + \mu_1 \partial + \mu_2 (\kappa \partial^2 + 2A \partial) \right) f + f^{-1} \bar{A} f. \quad (3.18)$$

The constraints (3.12) generating (3.18) are

$$\bar{\partial}A + \partial \left(\mu_1 A + \mu_2 (A^2 + \kappa \partial A) \right) - \kappa \partial \bar{A} + [\bar{A} - \mu_2 \partial A, A] = \sum_{a=1}^n p_a \delta(x_a). \quad (3.19)$$

The linear equation (3.14) takes the form

$$(\bar{\partial} + \mu_1 \partial + \mu_2 (\kappa \partial^2 + 2A \partial) + \bar{A}) \Psi = 0, \quad (3.20)$$

while (3.13) and (3.15) are the same. The operator in the left hand side defined generalized holomorphic structure on the bundle E .

3. Symplectic reduction. As for the complex structure we use the symplectic reduction to derive the nontrivial equations. Let fix \bar{A} similar to (2.20)

$$\bar{L} = f^{-1} \left(\bar{\partial} + \sum_{i=1}^j \mu_i \sum_{l=0}^{i-1} \kappa^{i-l-1} C_i^l \tilde{A}^{(l)} \partial^{i-l} \right) f + f^{-1} \bar{A} f, \quad (3.21)$$

where again $\bar{L} = 0$ for $g = 0$, \bar{L} =diagonal constant matrix for $g = 1$, and antiholomorphic for an arbitrary genus curves. Substituting \bar{L} and L in (3.12) one obtains

$$\bar{\partial}L + \sum_{j=1}^k \partial(\mu_j \tilde{L}^{(j)}) - \kappa \partial \bar{L} + [\bar{L} - \frac{1}{\kappa} \sum_{j=1}^k \mu_j \tilde{L}^{(j)}, L] - \sum_{a=1}^n p_a \delta(x_a) = 0, \quad (3.22)$$

where $\tilde{L}^{(j)}$ is defined as $\tilde{A}^{(j)}$

$$\tilde{L}^{(j)} = L^j + \kappa + \sum_{m=1}^{j-1} m L^{m-1} \partial L L^{j-m-1} \dots + \kappa^{j-1} \partial^{j-1} L.$$

The symplectic reduction leads to the same phase space $\mathcal{R}_{red}^N = \{L, \bar{L}, \mathbf{p}\}$. It defines the bundle $\mathcal{P}_k^{red,N}$ over $\mathcal{T}(k)_{g,n}$ and

$$\dim \mathcal{P}_k^{red,N} = \left(2(N^2 - 1) + (k + 1)^2 - 1\right) (g - 1) + (N^2 - N + \frac{1}{2}k(k + 1))n.$$

The form ω (3.7) on $\mathcal{P}_k^{red,N}$

$$\omega = \int_{\Sigma_{g,n}} \langle dL \, d\bar{L} \rangle - \frac{1}{\kappa} \int_{\Sigma_{g,n}} \langle dL \, d(\sum_{j=1}^k \tilde{L}^{(j)}) \rangle \sum_{s=1}^{l_j} t_{s,j} \mu_{s,j}^{(0)} + \omega_{\mathcal{O}}.$$

is degenerate as on \mathcal{P}_k^N . But now the equations of motion are no longer free. Nevertheless, after the symplectic reduction (3.9) and (3.10) take the form of the Lax equations

$$\partial_{s,j} L - \kappa \partial M_{s,j} + [M_{s,j}, L] = 0, \quad (s = 1, \dots, l_j; j = 1, \dots, k), \quad (3.23)$$

$$\bar{\partial} M_{s,j} + [M_{s,j}, \bar{L} - \frac{1}{\kappa} \sum_{j=1}^k \tilde{L}^{(j)} \sum_{s=1}^{l_j} \mu_{s,j}] = \kappa \partial_{s,j} \bar{L} - \sum_{j=1}^k \sum_{s=1}^{l_j} (\tilde{L}^{(j)} \mu_{s,j}^0 + \mu_{s,j}^0 t_{s,j} \partial_{s,j} \tilde{L}^{(j)}) \quad (3.24)$$

As before, these equations along with the constraints (3.22) are consistency conditions for the linear system

$$(\kappa \partial + L) \Psi = 0, \quad (3.25)$$

$$(\bar{\partial} + \frac{1}{\kappa} \sum_{j=1}^k \mu_j \sum_{m=0}^{j-1} C_j^m \tilde{L}^{(m)} (\kappa \partial)^{j-m} + \bar{L}) \Psi = 0, \quad (3.26)$$

$$(\kappa \partial_{s,j} + M_{s,j}) \Psi = 0, \quad (\partial_{s,j} = \partial_{t_{s,j}}). \quad (3.27)$$

4 Integrable hierarchies in the scaling limit

In the limit $\kappa \rightarrow 0$ the Hamiltonian form of equations is restored. We come to an autonomous hierarchy after the same redefinition of the times

$$t_{s,j} = t_{s,j}^0 + \kappa t_{s,j}^H, \quad (4.1)$$

where $t_{s,j}^0$ are fixed and $t_{s,j}^H$ are the genuine times. We fix a point in the generalized Teichmüller space $\mathcal{T}(k)_{g,n}$

$$\mu_j(0) = \sum_{s=1}^{l_j} t_{s,j}^0 \mu_{s,j}^0, \quad (j = 1, 2, \dots, k) \quad (4.2)$$

1. Scaling limit in the upstairs systems. Consider the "quasi-classical limit" of the linear problem for the isomonodromy hierarchy (3.13), (3.14), and (3.15). Put as before (2.31)

$$\Psi = \phi \exp \frac{\mathcal{S}}{\kappa}.$$

To come to the isospectral problem for Φ in (3.13) we define

$$\partial \mathcal{S} = \lambda. \quad (4.3)$$

To exclude the singular terms of order κ^{-1} in (3.14) and (3.15) we assume that the phase \mathcal{S} satisfies in addition to (4.3)

$$\begin{aligned} (\partial_{\bar{w}_0} - \sum_{j=2}^k (-1)^j \mu_j(0) \lambda^j) \mathcal{S} &= 0, \\ \frac{\partial}{\partial t_{s,j}^H} \mathcal{S} &= 0. \end{aligned} \quad (4.4)$$

Let $\Phi = \lim_{\kappa \rightarrow 0} A$ be the Higgs field $\Phi \in \Omega^{(1,0)}(\Sigma_{g,n}, \mathfrak{sl}(N, \mathbf{C}))$. As before, introduce the matrix Baker-Akhiezer function

$$Y = \phi \exp \sum_{s,j} t_{s,j}^H \frac{\partial \mathcal{S}}{\partial t_{s,j}^0}.$$

Then in the limit $\kappa \rightarrow 0$ (3.13), (3.14), and (3.15) take the form of the isospectral problem

$$(\lambda + \Phi)Y = 0, \quad (4.5)$$

$$(\bar{\partial} + \mu_1^0 \partial + \sum_{j=1}^k \mu_j(0) S_j(\Phi, \partial \Phi) - \sum_{j=1}^k \sum_{s=1}^{l_j} t_{s,j}^H \mu_{s,j}^0 \Phi^j + \bar{A})Y = 0, \quad (4.6)$$

$$\partial_{s,j} Y = 0. \quad (4.7)$$

Here the field S_j is a polynomial in Φ with a linear dependence on $\partial \Phi$. It is defined as a result of the action of the differential operator in (3.14) on Ψ (2.31) in the limit $\kappa \rightarrow 0$

$$S_j(\Phi, \partial \Phi) = \lim_{\kappa \rightarrow 0} \frac{1}{\kappa} \sum_{s=1}^{l_j} (t_{s,j}^0 + t_{s,j}^H) \sum_{m=0}^{j-1} C_j^m \tilde{A}^{(m)}(\kappa \partial)^{j-m} \left(\phi \exp \frac{\mathcal{S}}{\kappa} \right). \quad (4.8)$$

The explicit form $S_j(\Phi, \partial \Phi)$ for small j is

$$S_1(\Phi, \partial \Phi) = 0, \quad S_2(\Phi, \partial \Phi) = -2\partial \Phi, \quad S_3(\Phi, \partial \Phi) = 3\Phi \partial \Phi, \quad (4.9)$$

$$S_4(\Phi, \partial \Phi) = -4\Phi^2 \partial \Phi + \partial \Phi \Phi^2 + 6\Phi \partial \Phi \Phi.$$

Consider the consistency conditions for (4.5), (4.7) and (4.6). (4.7)

$$\partial_{s,j} \Phi = 0, \quad \partial_{s,j} \bar{A} = \Phi^j \mu_{s,j}^0, \quad (\partial_{s,j} = \frac{\partial}{\partial t_{s,j}^H}). \quad (4.10)$$

They can be interpreted as the equations of motion of the Hamiltonian system on \mathcal{R}^N equipped by the non degenerate symplectic form $\omega^H + \omega_{\mathcal{O}}$

$$\omega^H = \int_{\Sigma_{g,n}} \langle d\Phi d\bar{A}' \rangle + \omega_{\mathcal{O}}, \quad \bar{A}' = \sum_{j=1}^k \mu_j(0) S_j(\Phi, \partial \Phi) + \bar{A}, \quad (4.11)$$

and the set of Hamiltonians

$$H_{s,j} = \frac{1}{j+1} \int_{\Sigma_{g,n}} < \Phi >^{j+1} t_{s,j}^0 \mu_{s,j}^0, \quad (j = 1, \dots, k).$$

Then the equations of motion for this Hamiltonian hierarchy coincide with (4.10) along with

$$\partial_{s,j} p_a = 0, \quad \partial_{s,j} g_a = 0.$$

In particular, the Hamiltonian $H_{s,j}$ are in involution.

2.Symplectic reduction. The gauge symmetries of this system is

$$\Phi \rightarrow f^{-1} \Phi f, \quad \bar{A}' \rightarrow f^{-1} \partial_{\bar{w}_0} f + f^{-1} \bar{A}' f. \quad (4.12)$$

The orbit degrees of freedom are transformed as before (2.4). The moment constraints imposed by these symmetries take the form

$$\partial_{\bar{w}_0} \Phi + \sum_{j=2}^k \mu_j(0) [S_j(\Phi, \partial \Phi), \Phi] + [\bar{A}, \Phi] = 2\pi i \sum_{a=1}^n \delta(x_a) p_a. \quad (4.13)$$

For $k = 2$, $\bar{A}' = \bar{A} - 2\mu_2(0) \partial \Phi$,

$$\omega^H = \int_{\Sigma_{g,n}} < d\Phi d\bar{A} > - 2\mu_2 < d\Phi \partial d\Phi >,$$

the gauge transformations of \bar{A}

$$\bar{A} \rightarrow f^{-1} \left(\bar{\partial}_{\bar{w}_0} - 2\mu_2(0) ([\Phi, \partial f f^{-1}] + \partial \Phi) \right) f + f^{-1} \bar{A} f,$$

and the moment constraints take the form

$$\partial_{\bar{w}_0} \Phi + [\bar{A} - 2\mu_2(0) \partial \Phi, \Phi] = 2\pi i \sum_{a=1}^n \delta(x_a) p_a. \quad (4.14)$$

In general case, after the symplectic reduction we come to the generalized Hitchin equation (GHE) depending on parameters $\mu_j(0) \in \mathcal{T}(k)_{g,n}$

$$\bar{\partial}_{\bar{w}_0} L + \sum_{j=2}^k \mu_j(0) [S_j(L, \partial L), L] + [\bar{L}, L] = 2\pi i \sum_{a=1}^n \delta(x_a) p_a, \quad (4.15)$$

where \bar{L} is assumed to be fixed by the gauge transformations (4.12) and $L = f^{-1} \Phi f$. The solutions of (4.15) determine an autonomous Hamiltonian system on the reduced phase space $\mathcal{R}_{red}^N = (L, \bar{L}, \mathbf{p})$. The symplectic form on \mathcal{R}_{red}^N is

$$\omega^H = \int_{\Sigma_{g,n}} < \delta L \delta \bar{L}' > + \omega_{\mathcal{O}}, \quad \bar{L}' = \bar{L} - \sum_{j=2}^k \mu_j(0) S_j(L, \partial L) \quad (4.16)$$

and

$$H_{s,j} = \frac{1}{j+1} \int_{\Sigma_{g,n}} < L^{j+1} > t_{s,j}^0 \mu_{s,j}^0.$$

Since $H_{s,j}$ are in involution on \mathcal{R}^N the same is true after the symplectic reduction on \mathcal{R}_{red}^N . Assume now that $k+1 = N$. Then the number of the integrals is equal to $\dim \mathcal{T}(k)_{g,n} = \frac{1}{2} \dim \mathcal{R}^N$ (see (3.4) and (2.23)). Thereby, the limiting autonomous system is completely integrable. The fixed times $t_{s,j}^0$ can be considered as the deformation parameters of the integrable system on the phase space $\mathcal{R}^N = (L_0, \bar{L}_0, \mathbf{p})$.

The accompanying linear problem takes the form

$$\begin{aligned} (\lambda + L)Y &= 0, \\ (\bar{\partial}_{\bar{w}_0} + \sum_{j=1}^k \mu_j(0)S_j(L, \partial L) + \bar{L})Y &= 0, \\ (\partial_{t_{s,j}^H} + M_{0,s,j})Y &= 0. \end{aligned}$$

The first equation allows to construct the spectral curve

$$\mathcal{C}: \det(\lambda + L(w_0, \mu_2(0), \dots, \mu_k(0))) = 0.$$

The phase \mathcal{S} in (2.31) as before gives rise to the Seiberg-Witten differential (4.3) on \mathcal{C} . But in contrast with the case $k = 1$ it is no longer meromorphic in the complex structure (w_0, \bar{w}_0) on $\Sigma_{g,n}$ (4.4). The reason is that in general, if $k > 1$, then complex structures on \mathcal{C} cannot be described in terms of complex structures on $\Sigma_{g,n}$.

3. Solutions of GHE via perturbation expansions. GHE (4.15) is a nonlinear differential matrix equation (for $k > 1$). The first nontrivial case is $k = 2$

$$\partial_{\bar{w}_0} L + [\bar{L} - 2\mu_2(0)\partial L, L] = 2\pi i \sum_{a=1}^n \delta(x_a) p_a.$$

It looks hopeless to find explicit solutions of GHE as it was done for $k = 1$ [5]. At the moment the only way is to construct the series expansions assuming that the deformation parameters $t_{s,j}^0$ are small. Therefore, we are investigating only a neighborhood $\mu_j = 0$, $j = 2, \dots, k$ in the space of generalized complex structures $\mathcal{T}(k)_{g,n}$.

Consider deformations in a one specific direction $\mu_{s,j}^0$ and put $\varepsilon = t_{s,j}^0$. Let L_0 satisfies the usual Hitchin equation (2.35) and represent L as

$$L = L_0 + \sum_n \varepsilon^n L_n.$$

Substituting this expansion in GHE one obtain the chain of recurrence differential relations

$$\partial_{\bar{w}} L_n + [\bar{L}, L_n] = \mu_{s,j}^0 [L_{n-1}, S_j(L_{n-1}, \partial L_{n-1})]. \quad (4.17)$$

Let $g = 0$. Then, as we have already mentioned, $\bar{L} = 0$ and (4.17) splits in the set of differential equations

$$\partial_{\bar{w}} (L_n)_{ml} = (F_{n-1})_{ml}, \quad (m, l = 1, \dots, N),$$

where

$$F_{n-1} = \mu_{s,j}^0 [L_{n-1}, S_j(L_{n-1}, \partial L_{n-1})],$$

The solutions can be expressed in the integral form

$$(L_n(w, \bar{w}))_{ml} = \int_{\Sigma_{g,n}} G(w - z, \bar{w} - \bar{z}) (F_{n-1}(z, \bar{z}))_{ml}, \quad (4.18)$$

where G is a Green function on a rational curve

$$G = \frac{1}{w - z}, \quad \partial_{\bar{w}} G = \delta(z, \bar{z}).$$

Consider next the case $g = 1, n = 1$. For elliptic curves \bar{L} can be diagonalized $\bar{L} = \text{diag}(u_1, \dots, u_N)$ and thereby (4.17) again splits. Consider, for example, the non-diagonal part of (4.17)

$$\partial_{\bar{w}}(L_n)_{ml} + (u_m - u_l)(L_n)_{ml} = (F_{n-1})_{ml}, \quad (m \neq l, \quad m, l = 1, \dots, N).$$

Then again we represent $(L_n)_{ml}$ as (4.18), where the Green function of this equation has the form [5]

$$G(z, \bar{z}) = \exp\left(\frac{z - \bar{z}}{\tau - \bar{\tau}}(u_m - u_l)\right) \frac{\theta(u_m - u_l + z)\theta'(0)}{\theta(u_m - u_l)\theta(z)}.$$

In some cases it is possible to calculate the integrals explicitly, and thereby to find the corrections to the Lax operator induced by small perturbations of the generalized complex structures. The detail analysis of corresponding Hamiltonian systems will be published elsewhere.

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